AFFINE FUNCTIONS ON SIMPLEXES AND EXTREME OPERATORS

BY

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ABSTRACT

If K is a simplex and X a Banach space then $A(K, X)$ denotes the space of affine continuous functions from K to X with the supremum norm. The extreme points of the closed unit ball of $A(K, X)$ are characterized, X being supposed to satisfy certain conditions. This characterization is used to investigate the extreme compact operators from a Banach space X to the space $A(K) = A(K, (-\infty, \infty))$.

1. If S is a compact Hausdorff space then it is well known that $f \in C(S)$ is an extreme point of the closed unit ball if and only if $|f(s)| = 1$ everywhere on S. Our first aim is to extend this characterization to the more general situation of the space $A(K,X)$ — the space of affine continuous functions on the simplex K having values in the Banach space X with the supremum norm: $\|f\| = \sup_{k \in K}\|f(k)\|$. It is shown that if X is strictly convex or if every three mutually intersecting closed balls of X have a point in common then $f \in A(K,X)$ is an extreme point of the closed unit ball if and only if it maps the extreme points of K into the extreme points of the closed unit ball of X (Theorem 3.4). We obtain this characterization through a similar one for the extreme points of the closed unit ball of $A^*(K,X)$ (Theorem 3.2). In Section 2 we discuss the maximal convex subsets of the unit sphere of $A(K) = A(K,(-\infty, \infty))$; their simple representation is helpful in the next section. In Section 4 we use our results to investigate the extreme compact operators from a Banach space X into $A(K)$. This section contains also a characterization of the extreme positive operators from a $C(S)$ space (S metrizable compact Hausdorff) to $A(K)$: their representing functions map the extreme points of K into the point measures of S (Theorem 4.2). The paper ends with an example which shows that the extreme positive operators from an $A(K)$ space to a $C(S)$ space cannot be characterized in a similar manner.

We deal only with Banach spaces over the real field. The closed unit ball of a Banach space X is denoted by S_X. A Banach space is said to have the n.2. intersection property (n.2.I.P.) if every collection of n mutually intersecting closed

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balls in X has a common point. By operators we always mean a bounded linear operator.

If K is a convex subset of a linear space then $F \subset K$ is a face of K if it is convex and satisfies: $0 < \lambda < 1$, $k_1, k_2 \in K$, $\lambda k_1 + (1-\lambda)k_2 \in F \Rightarrow k_1, k_2 \in F$. An extreme point of K is a one point face of it. The set of the extreme points of K is denoted by ∂K.

Let K be a compact convex subset of a locally convex linear topological space. The probability Radon measures on K are ordered by

$$\mu \succ \mu_2 \Leftrightarrow \int_K \phi d\mu_1 \geqq \int_K \phi d\mu_2$$

for every convex continuous function ϕ on K (Choquet's ordering). For each $k \in K$ there is a measure μ on K, maximal in this ordering, which represents k, that is, $f(k) = \int_K f d\mu$ for any affine continuous function f on K (cf. [4], [15]). K is a simplex if for each of its points the representing maximal measure is unique. The reader may find all the fundamental facts about simplexes in [4] or [15].

If S is a compact Hausdorff space the probability Radon measures on S form a simplex in $C^*(S)$ when this space is endowed with its w^*-topology. This simplex is of a special type since its extreme points form a closed set. $C(S)$ is isometrically isomorphic with the space of all the affine continuous functions on this simplex. We shall make no distinction between S and its canonic image in $C^*(S)$. Similarly, any simplex K can be imbedded by an affine homeomorphism into $A^*(K)$: $Tk(f) = f(k)$ for $k \in K$, $f \in A(K)$. It is convenient to consider K imbedded in this way into $A^*(K)$.

A collection of non-negative functions $\{f_i\}_{i=1}^n \subset A(K)$ is called a partition of the unity on the simplex K if $\sum_{i=1}^n f_i = 1$.

If M is a set then we denote by 2^M the family of all its subsets. Let X, Y be topological spaces. A map $T: X \to 2^Y$ is said to be lower semi-continuous if for any open set $U \subset Y$ the set $\{x \in X : T(x) \cap U \neq \emptyset\}$ is open in X. Suppose that E and F are linear spaces and K is a convex subset of E. A map $T: K \to 2^F$ is called affine if $T(k)$ is a non-void convex subset of F for every $k \in K$ and

$$\lambda T(k_1) + (1-\lambda) T(k_2) \subset T(\lambda k_1 + (1-\lambda)k_2)$$

when $0 < \lambda < 1$, $k_1, k_2 \in K$. The following theorem on multi-valued maps defined on simplexes was proved in [12] and it is stated here for the convenience of the reader.

THEOREM 1.1. (*cf.* [12, *Theorem* 3.1, *Corollary* 3.3]).
Let E be a Frechet space and K a simplex. Suppose that $T: K \to 2^E$ is an affine lower semi-continuous map such that $T(k)$ is closed for every $k \in K$. Then there exists an affine continuous selection for T, that is, an affine continuous

function $f: K \to E$ *with* $f(k) \in T(k)$ *for each* $k \in K$. *Moreover, if* $k_0 \in \partial K$ *and* $x \in T(k_0)$ *then* f *can be chosen such that* $f(k_0) = x$.

2. The following theorem was proved by Eilenberg [9] for the space of all continuous functions on a compact Hausdorff space. The proof given below is an adaptation of his proof to the more comprehensive class of the spaces of affine continuous functions on simplexes.

THEOREM 2.1. *Let* K *be a simplex and* Q *a maximal convex subset of* $\{f \in A(K): \|f\| = 1\}$. *Then there exist* $k_0 \in \partial K$ *and a sign* ε *(i.e.* $\varepsilon = \pm 1$*) such that*

$$(1) \qquad Q = \{f \in A(K): f(k_0) = \varepsilon, \ \|f\|\}.$$

Conversely, every set determined by a point $k_0 \in \partial K$ and a sign ε as in (1) is a maximal convex subset of the boundary of $S_{A(K)}$.

Proof. Let Q be a maximal convex subset of $\{f \in A(K): \|f\| = 1\}$. For every $f \in Q$ we define the following closed faces of K:

$$F_f^+ = \{k \in K: f(k) = 1\}, \ F_f^- = \{k \in K: f(k) = -1\}.$$

The first assertion of the theorem will be proved if we show that $\cap_{f \in Q} F_f^+ \neq \emptyset$. or $\cap_{f \in Q} F_f^- \neq \emptyset$. Indeed, if one of these sets is non-void, say $\cap_{f \in Q} F_f^+ \neq \emptyset$, then, being a closed face of K, we can find $k_0 \in \cap_{f \in Q}(F_f^+ \cap \partial K)$. Clearly

$$(2) \qquad Q \subset \{f \in A(K): f(k_0) = \|f\| = 1\}.$$

and the maximality of Q implies that (2) holds with equality sign between its members.

Let us suppose that $\cap_{f \in Q} F_f^+ = \cap_{f \in Q} F_f^- = \emptyset$. By the compactness of K there are $\{f_i\}_{i=1}^n, \{g_j\}_{j=1}^m \subset Q$ such that

$$(3) \qquad \bigcap_{i=1}^{n} F_{f_i}^+ = \bigcap_{j=1}^{m} F_{g_j}^- = \emptyset.$$

Since Q is convex we have $(\sum_{i=1}^n f_i + \sum_{j=1}^m g_j)/(m+n) \in Q$. Hence $\| \sum_{i=1}^n f_i + \sum_{j=1}^m g_j \| = m + n$. Obviously this equality contradicts (3) so (1) holds for a certain point $k_0 \in \partial K$.

Now let $k_0 \in \partial K$, $\varepsilon = \pm 1$ and

$$Q = \{f \in A(K): f(k_0) = \varepsilon, \ \|f\| = 1\}.$$

Assume that there is a convex subset $Q' \subset \{f \in A(K): \|f\| = 1\}$ such that $Q' \supset Q$, $Q' \neq Q$. Pick $f \in Q' \sim Q$ and denote

$$F^+ = \{k \in K: f(k) = 1\}, \ F^- = \{k \in K; f(k) = -1\}.$$

The extreme point k_0 does not belong to the closed face $F = \operatorname{conv}(F^+ \cup F^-)$

since $k_0 \notin F^+ \cup F^-$. Indeed, $k_0 \in F^+ \cup F^-$ implies $f \in Q \cup (-Q)$ and if this were true then $0 = \frac{1}{2}[f + (-f)]$ would belong to Q. By a theorem of Edwards [8] there exists $f' \in A(K)$ such that $f'_{|F} = 0$, $f'(k_0) = \varepsilon$ and $\|f'\| = 1$. Clearly $\|f + f'\| < 2$ therefore $\frac{1}{2}(f + f') \notin Q'$ so Q' cannot be convex. This concludes the proof of the theorem.

From a theorem of Lindenstrauss [13, Theorem 4.8] it follows that if Q is a maximal convex subset of $\{f \in A(K) : \|f\| = 1\}$ then $S_{A(K)} = \overline{\text{conv}}(Q \cup (-Q))$. We are going to prove that the closure is superfluous here. Of course, this is well-known for spaces of continuous functions (see [11]).

THEOREM 2.2. *If K is a simplex and Q is a maximal convex subset of $\{f \in A(K): \|f\| = 1\}$ then $S_{A(K)} = \text{conv}(Q \cup (-Q))$.*

Proof. We have to show that $S_{A(K)} \subset \text{conv}(Q \cup (-Q))$. Without loss of generality we may suppose that there exists a $k_0 \in \partial X$ such that

$$Q = \{f \in A(K) : f(k_0) = \|f\| = 1\}.$$

Let $f \in S_{A(K)}$. If $f \in Q \cup (-Q)$ there is nothing to prove so we may assume that $f \notin Q \cup (-Q)$. Define the following affine continuous functions on K:

$$f_1(k) = \frac{2f(k) - 1 + f(k_0)}{1 + f(k_0)}, \quad f_2(k) = \frac{2f(k) + 1 - f(k_0)}{1 + f(k_0)}, \quad k \in K.$$

It is easy to check that $h_1 = f_1 \vee (-1) \leqq f_2 \wedge 1 = h_2$ and $h_2(k_0) = 1$. From Edwards' separation theorem [8] we infer that there exists a $g_1 \in A(K)$ such that $h_1 \leqq g_1 \leqq h_2$ and $g_1(k_0) = 1$. If

$$g_2(k) = \frac{2f(k) - (1 + f(k_0))g_1(k)}{1 - f(k_0)}, \quad k \in K,$$

then it is clear that $g_2 \in -Q$ and

$$f = \tfrac{1}{2}[(1 + f(k_0))g_1 + (1 - f(k_0))g_2].$$

Since $-1 < f(k_0) < 1$ we proved that $f \in \text{conv}(Q \cup (-Q))$ and this concludes the proof of the theorem.

3. We now pass to the space $A(K,X)$ and its dual. The following lemma is an easy consequence of Lemma 2.4 of [12].

LEMMA 3.1. *Let K be a simplex and X a Banach space. The following subset of $A(K,X)$ is norm dense in $A(K,X)$:*

$$\left\{ \sum_{i=1}^{n} \phi_i x_i : \{x_i\}_{i=1}^{n} \subset X,\ \{\phi_i\}_{i=1}^{n} \subset A(K),\ \sum_{i=1}^{n} \phi_i = 1,\ \phi_i \geqq 0 \right\}.$$

THEOREM 3.2. *Let K be a simplex, X a Banach space, $k \in \partial K$ and $x^* \in \partial S_{X^*}$. The functional $y^*_{k,x^*} \in A^*(K,X)$ defined by*

$$y^*_{k,x^*}(y) = x^*(y(k)), \quad y \in A(K,X) \tag{1}$$

is an extreme point of the closed unit ball of $A^*(K,X)$. Conversely, to every extreme point of this ball there correspond a $k \in \partial K$ and an $x^* \in \partial S_{X^*}$ related to it by (1).

Proof. Denote $Y = A(K,X)$. Clearly if $k \in \partial K$ and $x^* \in \partial S_{X^*}$ then the functional y^*_{k,x^*} given by (1) belongs to S_{Y^*}. Suppose that there are $y^*_1, y^*_2 \in S_{Y^*}$ such that

$$y^*_{k,x^*} = \tfrac{1}{2}(y^*_1 + y^*_2), \qquad y^*_1 \neq y^*_{k,x^*}. \tag{2}$$

From the preceding lemma we infer the existence of a partition of unity on K, $\{\phi_i\}_{i=1}^n \subset A(K)$ and the existence of points $\{x_i\}_{i=1}^n \subset X$ for which

$$y^*_1\left(\sum_{i=1}^n \phi_i x_i\right) \neq y^*_{k,x^*}\left(\sum_{i=1}^n \phi_i x_i\right).$$

Then there is an index i, $1 \leqq i \leqq n$, such that $y^*_1(\phi_i x_i) \neq y^*_{k,x^*}(\phi_i x_i)$. By Theorem 2.1 and Theorem 2.2 there is a $\psi \in A(K)$ which satisfies:

$$\psi(k) = \|\psi\| = 1, \; y^*_1(\psi x_i) \neq y^*_{k,x^*}(\psi x_i).$$

Define two functionals $x^*_1, x^*_2 \in S_{X^*}$ by

$$x^*_1(x) = y^*_1(\psi x), \; x^*_2(x) = y^*_2(\psi x), \qquad x \in X.$$

From (2) it follows that

$$\tfrac{1}{2}(x^*_1 + x^*_2)(x) = y^*_{k,x^*}(\psi x) = x^*(\psi(k)x) = x^*(x),$$

that is, $x^* = \frac{1}{2}(x^*_1 + x^*_2)$. Hence $x^* = x^*_1$ and in particular

$$y^*_1(\psi x_i) = x^*_1(x_i) = x^*(x_i) = y^*_{k,x^*}(\psi x_i).$$

We obtained a contradiction and by this the first part of the theorem is proved.

Now we pass to show that any extreme point of S_{Y^*} can be represented as in (1). Let $\mathscr{E} = \{y^*_{k,x^*} : k \in \partial K, x^* \in \partial S_{X^*}\}$. First we prove that ∂S_{Y^*} is included in the weak* closure of $\mathscr{E}$. To see this it suffices to show that S_{Y^*} is the weak* closure of conv$\mathscr{E}$ (cf. [7, p. 80]). Let $y^*_0 \in S_{Y^*}$ and suppose that $y_0 \notin w^* - cl$ (conv $\mathscr{E}$). Then, by the separation theorem for compact convex sets there exist a $y \in Y$ and a real number α such that $y^*_0(y) > \alpha$ and $y^*(y) < \alpha$ for every $y^* \in w^* - cl(\text{conv}\,\mathscr{E})$. In particular

$$x^*(y(k)) = y^*_{k,x^*}(y) < \alpha, \; k \in K, \; x^* \in \partial S_{X^*}.$$

By the Krein-Milman theorem and Bauer's maximum principle [2] it follows that $\|y\| \leqq \alpha$ in contradiction with $y_0^*(y) > \alpha$. Consequently, $\partial S_{Y^*} \subset w^* - cl(\mathscr{E})$.

Let $y^* \in S_{Y^*}$. By what we have just proved we can find two nets: $\{k_i\}_{i \in I} \subset \partial K, \{x_i^*\}_{i \in I} \subset \partial S_{X^*}$ such that $\{y^*_{k_i,x^*_i}\}_{i \in I}$ converges to y^* in the w^*-topology of Y^*. We may assume that the first net converges to $k \in K$ and the second converges to $x^* \in S_{X^*}$ in the w^*-topology of X^*. Define $y^*_{k,x^*} \in S_{Y^*}$ by

$$y^*_{k,x}(y) = x^*(y(k)), \qquad y \in Y.$$

We have

$$\begin{aligned} \left|y^*_{k,x^*}(y) - y^*_{k_i,x_i^*}(y)\right| &\leqq \left|x^*(y(k)) - x_i^*(y(k))\right| + \\ + \left|x_i^*(y(k)) - x_i^*(y(k_i))\right| &\leqq \left|x^*(y(k)) - x_i^*(y(k))\right| + \|y(k) - y(k_i)\|. \end{aligned}$$

It is easily seen from the above inequality that $y^* = y^*_{k,x^*}$. Clearly $y^* \in \partial S_{Y^*}$ implies that $k \in \partial K$, $x^* \in \partial S_{X^*}$. This concludes the proof of the theorem.

Now we turn to the space $A(K,X)$ itself. The following theorem generalizes a result of Lindenstrauss [13, p. 43].

THEOREM 3.3. *Let X be a Banach space having the n.2.I.P. ($n \geqq 3$) and K a simplex. Then $A(K,X)$ has the n.2.I.P.*

Proof. According to [13, Lemma 4.2] it is enough to show that for any finite set $\{y_i\}_{i=1}^n \subset A(K,X)$ and any $\varepsilon > 0$ there exists a subspace $Z \subset A(K,X)$ having the n.2.I.P. such that the distance between y_i ($1 \leqq i \leqq n$) and Z is not greater than ε.

From [12, Lemma 2.4] we infer the existence of a partition of the unity on K, $\{\phi_j\}_{j=1}^m \subset A(K)$, $\|\phi_j\| = 1$, $1 \leqq j \leqq m$ and the existence of a set $\{x_{ij} : 1 \leqq i \leqq n,\ 1 \leqq j \leqq m\} \subset X$ for which

$$\left\| y_i(k) = \sum_{j=1}^m \phi_j(k) x_{ij} \right\| < \varepsilon, \quad k \in K,\ 1 \leqq i \leqq n. \tag{1}$$

It is easily seen that the subspace $Z \subset A(K,X)$,

$$Z = \left\{ \sum_{j=1}^m \phi_j x_j : \{x_j\}_{j=1}^m \subset X \right\},$$

is isometrically isomorphic with $(X \oplus X \oplus \cdots \oplus X)_{l_\infty^m}$. Hence Z has the n.2.I.P. (cf. [13, Theorem 4.6]). By (1) we know that the distance of y_i from Z is at most ε and this establishes the theorem.

THEOREM 3.4. *Let K be a simplex and X a Banach space. Assume that*

a) *X has the n.2.I.P. ($n \geqq 3$);*

or

b) *X is strictly convex.*

Then a function $y \in Y = A(K,X)$ *is an extreme point of* S_Y *if and only if* $y(k) \in \partial S_X$ *for every* $k \in \partial K$.

Proof. One implication is trivial. We prove only that the condition is necessary.

a) Let $y \in \partial S_Y$. Since Y has the n.2.I.P. then, according to [13, Theorem 4.7], we have $|y^*(y)| = 1$ for $y^* \in \partial S_{Y^*}$. Therefore, by Theorem 3.2, if $k \in \partial K$ and $x^* \in \partial S_{X^*}$ then $|x^*(y(k))| = 1$. Hence, if $k \in \partial K$, and $y(k) = \frac{1}{2}(x_1 + x_2)$, $x_1, x_2 \in S_X$ then for every $x^* \in \partial S_{X^*}$ we have $|x^*(x_1 + x_2)| = 2|x^*(y)| = 2$. It follows that $x^*(x_1) = x^*(x_2)$ for each $x^* \in \partial S_{X^*}$ and this together with the Krein-Milman theorem implies that $x_1 = x_2$.

b) We define the following map from S_X to 2^{S_X}

$$T(x) = \{x' \in S_X \colon \|2x - x'\| \leqq 1\} \quad , \quad x \in S_X .$$

It is obvious that $x \in T(x)$, $T(x)$ is closed and T is an affine map. We shall prove that it is also lower semi-continuous. We have to show that for any $x \in S_X$, any $x' \in T(x)$ and any sequence $\{x_n\}_{n=1}^{\infty}$ converging to x there are $x'_n \in T(x_n)$, $n = 1, 2, \cdots$ such that $\{x'_n\}_{n=1}^{\infty}$ converges to x'.

If $\|x\| = 1$ the above assertion is clear since in this case $T(x) = \{x\}$. Let $\|x\| < 1$. We choose a sequence of numbers $\lambda_n \in [0,1]$, $\lambda_n \to 1$, such that

$$\|x + \lambda_n(x' - x)\| \leqq 1 - \|x - x_n\|, \ \|x - \lambda_n(x' - x)\| \leqq 1 - \|x - x_n\| .$$

It is easy to check that $x_n + \lambda_n(x' - x) \in T(x_n)$ and $\|x' - [x_n + \lambda_n(x' - x)]\| \to 0$. This proves that T is lower semi-continuous.

Let us consider the map $T \circ y \colon K \to 2^X$ where $y \in \partial S_Y$. If for a certain $k \in \partial K$ we have $y(k) \notin \partial S_X$, that is $T \circ y(k) \neq \{y(k)\}$ then, according to Theorem 1.1, there is an affine continuous selection of $T \circ y$, y' say, for which $y'(k) \neq y(k)$. Since

$$y = \frac{y' + (2y - y')}{2}, \ y' \in S_Y, \ 2y - y' \in S_Y, \ y' \neq y,$$

we obtained the desired contradiction.

Remark. The conclusion of the previous theorem will no longer hold if the space X does not satisfy certain conditions like those imposed above. In [3] is given an example of a four-dimensional Banach space X such that not all the extreme points of the closed unit ball of $C([0,1],X)$ admit the representation expressed by Theorem 3.4.

4. The following lemma, stated also in [12, Lemma 4.1], gives a representation for operators having the range in $A(K)$.

Lemma 4.1. *Let* K *be a simplex,* X *a Banach space and suppose that* T *is an operator from* X *into* $A(K)$. *Then there exists an affine and* w^**-continuons function* $\chi \colon K \to X^*$ *such that*:

(1) $Tx(k) = \chi(k)(x), \quad x \in X, \ k \in K$,

(2) $\| T \| = \sup_{k \in K} \| \chi(k) \|$.

Conversely, to any affine and w^*-continuous function from K into X^* there corresponds an operator $T: X \to A(K)$, given by (1) *whose norm satisfies* (2). *T is compact if and only if χ is continuous in the norm topology of X^*.*

Combining Theorem 3.4 with the preceding lemma we obtain a characterization of the extreme compact operators whose range is the space $A(K)$. If X, Y are Banach spaces we denote by $\mathscr{L}(X, Y)$ the space of compact operators from X to Y with the usual norm.

THEOREM 4.2. *Let K be a simplex and X a Banach space whose dual has the n.2.I.P. ($n \geqq 3$) or is a strictly convex space. The operator $T \in \mathscr{L}(X, A(K))$ is an extreme point of the closed unit ball of $\mathscr{L}(X, A(K))$ if and only if there exists an affine and norm continuous function $\chi: K \to S_{X^*}$ such that*

$$T(x)(k) = \chi(k)(x), \ x \in X, \ k \in K$$

and $\chi(k) \in \partial S_{X^*}$ whenever $k \in \partial K$.

REMARKS. As pointed out above this characterization is not valid for any Banach space X. However, the theorem applies to a wide range of spaces which comprises all the L_p $(1 < p < \infty)$ spaces since they are strictly convex, the L_1 spaces and those whose duals are L_1 spaces. The last categories of spaces enter here since they include spaces having the 3.2.I.P. (cf. [13, p. 44, Theorem 6.1]). For compact operators between two spaces of continuous functions on compact Hausdorff spaces the result was proved in [3].

Now we turn to the characterization of extreme positive operators from a $C(S)$ space to an $A(K)$ space. The extreme positive operators between two spaces of continuous functions (and even in more general situations) were characterized by A. and C. Ionescu Tulcea, Phelps [14] and Ellis [10] using methods which rely on the algebraic structure of the spaces. We found the idea of the proof of the next theorem in [3].

THEOREM 4.3. *Let K be a simplex, S a compact Hausdorff metrizable space and $\mathscr{L}_1$ the set of positive operators T from $C(S)$ to $A(K)$ which satisfy $T1 = 1$. Then the following statements are equivalent for an operator T from $C(S)$ to $A(K)$:*

(i) *T is an extreme point of $\mathscr{L}_1$;*

(ii) *There is a function $\chi: K \to C^*(S)$ which is affine and continuous in the w^*-topology of $C^*(S)$, such that*

$$T(f)(k) = \chi(k)(f), \ f \in C(S), \ k \in K$$

and which maps ∂K into S;

(iii) $T1=1$ *and for any* $f, g \in C(S)$, $T(f \vee g)$ *is the least upper bound of* Tf *and* Tg *in* $A(K)$.

Proof. (i) ⇒ (ii). Denote by $\mathscr{M}_1(S)$ the set of probability Radon measures on S and define $\Phi: \mathscr{M}_1(S) \to 2^{C^*(S)}$ in the following manner:

$$\Phi(\mu) = \{\mu' \in C^*(S): 2\mu \geqq \mu' \geqq 0\}, \; \mu \in \mathscr{M}_1(S).$$

We shall prove that Φ is a lower semi-continuous map when $\mathscr{M}_1(S)$ and $C^*(S)$ are equipped with the w^*-topology. We have to show that if $\mu \in \mathscr{M}_1(S), \mu' \in \Phi(\mu)$ and U is any neighborhood of μ' then there exists a neighborhood V of μ such that $\Phi(\nu) \cap U \neq \varnothing$ whenever $\nu \in V$.

Let

$$(1) \qquad U = \left\{\nu' \in C^*(S): \left| \int_S f_i d\nu' - \int_S f_i d\mu' \right| \leqq 1, \; f_i \in C(S), 1 \leqq i \leqq n\right\}.$$

Suppose that there is a net $\{\nu_\alpha\} \subset \mathscr{M}_1(S)$ converging to μ for which $\Phi(\nu_\alpha) \cap U = \varnothing$ for every α. From the Radon-Nikodym theorem we infer that there exists a Borel function g on S such that $0 \leqq g \leqq 2$ and $d\mu' = g d\mu$. Choose $g_1 \in C(S)$, $0 \leqq g_1 \leqq 2$, which satisfies:

$$(2) \qquad \int_S |g - g_1| d\mu \leqq (2\|f_i\|)^{-1}; \; 1 \leqq i \leqq n, \; \|f_i\| \neq 0.$$

If we define the measure ν'_α on S by $d\nu'_\alpha = g_1 d\nu_\alpha$ then $\nu'_\alpha \in \Phi(\nu_\alpha)$ and

$$(3) \qquad \lim_\alpha \int_S f_i d\nu'_\alpha = \lim_\alpha \int_S f_i g_1 d\nu_\alpha = \int_S f_i g_1 d\mu.$$

We have

$$(4) \qquad \left| \int_S f_i d\mu' - \int_S f_i d\nu'_\alpha \right| \leqq \left| \int_S f_i g d\mu - \int_S f_i g_1 d\mu \right| + \left| \int_S f_i g_1 d\mu - \int_S f_i d\nu'_\alpha \right|.$$

From (1)–(4) we deduce that ν'_α is eventually in U and this is the desired contradiction.

We now define another map $\Phi' = \mathscr{M}_1(S) \to 2^{\mathscr{M}_1(S)}$ as follows:

$$\Phi'(\mu) = \Phi(\mu) \cap \mathscr{M}_1(S), \; \mu \in \mathscr{M}_1(S).$$

It is easy to see that Φ' is affine and $\Phi'(\mu)$ is a w^*-closed subset of $\mathscr{M}_1(S)$ for every $\mu \in \mathscr{M}_1(S)$. We shall show that Φ' is lower semi-continuous too. Take $\mu \in \mathscr{M}_1(S)$, $\mu' \in \Phi'(\mu)$ and suppose that $\{\nu_\alpha\} \subset \mathscr{M}_1(S)$ is a net w^*-converging to μ. By the lower semi-continuity of Φ there are measures $\nu'_\alpha \in \Phi(\nu_\alpha)$ such that the net $\{\nu'_\alpha\}$ converges to μ'. Define

$$\nu''_\alpha = \begin{cases} \nu'_\alpha/\|\nu'_\alpha\|, & \|\nu'_\alpha\| \geqq 1, \\ [2(1-\|\nu'_\alpha\|)\nu_\alpha + \nu'_\alpha]/(2-\|\nu'_\alpha\|), & \|\nu'_\alpha\| < 1. \end{cases}$$

Clearly $\nu''_\alpha \geqq 0$, $2\nu_\alpha - \nu''_\alpha \geqq 0$ and $\|\nu''_\alpha\| = 1$, therefore $\nu''_\alpha \in \Phi'(\nu_\alpha)$. Since

$$\lim_\alpha \|\nu'_\alpha\| = \lim_\alpha \nu'_\alpha(1) = \mu'(S) = 1,$$

the net $\{\nu''_\alpha\}$ is w^*-converging to μ'. We proved that for any $\mu \in \mathscr{M}_1(S)$, any $\mu' \in \Phi'(\mu)$ and any net $\{\nu_\alpha\} \subset \mathscr{M}_1(S)$ w^*-converging to μ there are measures $\nu''_\alpha \in \Phi'(\nu_\alpha)$ w^*-converging to μ' i.e. Φ' is lower semi-continuous.

Let T be an extreme point of $\mathscr{L}_1$ and $\chi: K - C^*(S)$ the function representing it given by Lemma 4.1. Obviously $\chi(K) \subset \mathscr{M}_1(S)$. The map $\Phi' \circ \chi: K \to 2^{\mathscr{M}_1(S)}$ fulfills all the conditions of Theorem 1.1. If $\chi(k)$ does not belong to S for a certain $k \in \partial K$, that is $\Phi'(\chi(k)) \neq \{\chi(k)\}$ then there is an affine continuous selection χ' of $\Phi \circ \chi$ whose value at k is different from $\chi(k)$. The selection theorem may be used here since $\mathscr{M}_1(S)$ can be imbedded into a Fréchet space by the separability of $C(S)$ (see, for instance, the proof of Theorem 3.5 in [12]). If T' is the operator from $C(S)$ to $\mathscr{M}_1(S)$ corresponding to χ' then T' and $2T - T'$ belong to $\mathscr{L}_1$. This is a contradiction since T is an extreme point of $\mathscr{L}_1$.

The proof of (ii) $\Rightarrow$ (i) is trivial. We turn to (ii) = (iii). If (ii) holds then $T1 = 1$. Pick $f, g \in C(S)$. Obviously $T(f \vee g) \geqq Tf, Tg$. Let $h \in A(K)$, $h \geqq Tf, Tg$. If $k \in \partial K$ we have

$$\begin{aligned} T(f \vee g)(k) = (f \vee g)(\chi(k)) &= f(\chi(k)) \vee g(\chi(k)) \\ &= T(f)(k) \vee T(g)(k) \geqq h(k). \end{aligned}$$

By the maximum principle of Bauer [2] this implies $T(f \vee g) \geqq h$.

(iii) $\Rightarrow$ (ii). Let $\chi: K \to C^*(S)$ be the function representing the operator T given by Lemma 4.1. If $f, g \in C(S)$, $k \in \partial K$ then

$$\begin{aligned} (f \vee g)(\chi(k)) = (T(f) \vee T(g))(k) &= T(f)(k) \vee T(g)(k) \\ &= f(\chi(k)) \vee g(\chi(k)). \end{aligned}$$

This means that $\chi(k)$ is a lattice homomorphism of $C(S)$ into $(-\infty, \infty)$, which maps the function identically equal to 1 on S to 1. Hence, $\chi(k) \in S$ (cf. [7, p. 97]) and this completes the proof of the theorem.

Remark. The assumption of metrizability of S entered in the proof only through Theorem 1.1. Therefore, the conclusion of Theorem 4.2 is valid also if K is a metrizable simplex and S is homeomorphic with a w-compact subset of a Banach space (see [5], [6] and [1]). It is likely that the theorem is true without any restrictions on S or on K but we have not succeeded in proving it.

The situation is entirely different if we interchange the roles of the spaces $A(K)$ and $C(S)$ in the previous theorem. Let A be the space of the sequences $\{x_n\}_{n=1}^{\infty}$ converging to $\frac{1}{2}(x_1 + x_2)$ with the supremum norm. By [13, p. 78, Theorem 4.7] and [16] there is a simplex K such that $A = A(K)$. For instance, K may be the positive face of the unit ball of $l_1 = A^*$. Let T be the identity operator from A to c—the space of converging sequences. Then T is an extreme positive operator but the function from the compactification of the integers N_∞ to K representing it maps the unique non-isolated point of N_∞ to a non-extreme point of K. Still, a dense set of N_∞ is mapped into ∂K. We are going to show that for any compact Hausdorff space S there are a simplex K and an extreme positive operator T $T: A(K) \to C(S)$ such that the representing function of T maps $s \in S$ into ∂K if and only if s is an isolated point of S. A similar fact was proved in [3] but there the domain was not a space of affine continuous functions on a simplex.

Example 4.4. Let S be a compact Hausdorff space and S' the set of non-isolated points of S. Denote by $e_s (s \in S)$ the following function on S:

$$e_s(t) = \begin{cases} 0, & t \neq s, \\ 1, & t = s. \end{cases}$$

Obviously $e_s \in c_0(S)$, $e_s \in l_1(S)$. The dual of $X = (C(S) \oplus c_0(S))_{l^2_\infty}$ is $X^* = (C^*(S) \oplus l_1(S))_{l^2_1}$.

Consider the following subset of X^*:

$$M = \{(s, \pm e_s): s \in S'\} \cup \{(s,0): s \in S\}.$$

M is bounded and w^*-closed; thus $K = w^* - cl(\operatorname{conv} M)$ is a w^*-compact set whose extreme points belong to M. We shall show that K is a simplex but first we identify the extreme point of K. Clearly, if $s \in S'$ then $(s,0) \notin \partial K$. If $s \in S - S'$ then $(e_s, 0) \in X$ is a w^*-continuous linear functional on X^*. Its maximal value on K is 1 and it is attained only at $(s,0)$, thus $(s,0) \in \partial K$. Pick now $s \in S'$. The w^*-continuous linear functional $(0, e_s)$ takes its maximal value on M at (s, e_s) and its minimal value at $(s, -e_s)$. Consequently $(s, \pm e_s) \in \partial K$. We proved

$$\partial K = \{(s, \pm e_s): s \in S'\} \cup \{(s,0): s \in S - S'\}.$$

Now we turn to prove that K is a simplex. Let μ_1, μ_2 be two probability Radon measures on K maximal in the ordering of Choquet. That is, if μ is a positive Radon measure on K and $\int_K \phi d\mu \geqq \int_K \phi d\mu_i$ for every continuous convex function ϕ then $\mu = \mu_i$. Assume that $\int_K \psi d\mu_1 = \int_K \psi d\mu_2$ for each affine continuous function ψ. We have to show that $\mu_1 = \mu_2$.

We begin by showing that $\mu_1(\{(s,0)\}) = \mu_2(\{(s,0)\}) = 0$ if $s \in S'$. It suffices to carry on the proof only for μ_1. Suppose that this were not true and denote by $\varepsilon^+, \varepsilon^-, \varepsilon$ the point measures of (s, e_s), $(s, -e_s)$ and $(s,0)$, respectively. The measure

$$\mu = \mu_1 - \alpha\varepsilon + \frac{\alpha}{2}(\varepsilon^+ + \varepsilon^-),$$

where $\alpha = \mu_1(\{(s,0)\}) > 0$ is non-negative and if ϕ is a continuous convex function on K then

$$\int_K \phi(d\mu) = \int_K \phi d\mu_1 + \alpha[\tfrac{1}{2}(\phi(s,e_s) + \phi(s,-e_s)) - \phi(s,0)] \geqq \int_K \phi d\mu_1 .$$

Since μ_1 is maximal we have $\mu_1 = \mu$. Thus $\alpha = 0$ and our assertion is proved.

By a well-known property of maximal measures μ_1, μ_2 are concentrated on $\overline{\partial K}$ (cf. [4], [15, p. 30]), i.e., $\mu_1(M) = \mu_2(M) = 1$. Thus it is enough to prove the equality of their restrictions to M. The set $\{(s, \pm e_s): s \in S'\}$ contains only isolated points of M; therefore, if $E \subset \{(s, \pm e_s): s \in S'\}$ and if $a_s^i = \mu_i(\{(s,e_s)\})$ $b_s^i = \mu_i(\{(s,-e_s)\})$, then

$$\mu_i(E) = \Sigma\{a_s^i : (s,e_s) \in E\} + \Sigma\{b_s^i : (s,-e_s) \in E\}, \ i = 1,2.$$

Define two regular measures on the Borel sets of S by

$$m_i(T) = \mu_i(\{(s,0): s \in T\}), \ T \subset S, \ i = 1,2. \tag{1}$$

Let $f \in C(S)$, $f' \in c_0(S)$. Since $\int_K (f,f')d\mu_1 = \int_K (f,f')d\mu_2$ we have

$$\begin{aligned} &\int_S f dm_1 + \Sigma_{s \in S'} a_s^1(f(s) + f'(s)) + \Sigma_{s \in S'} b_s^1(f(s) - f'(s)) \\ &= \int_S f dm_2 + \Sigma_{s \in S'} a_s^2(f(s) + f'(s)) + \Sigma_{s \in S'} b_s^2(f(s) - f'(s)). \end{aligned} \tag{2}$$

We choose $f = 0$, $f' + e_s$ for $s \in S'$. From (2) we get

$$a_s^1 - b_s^1 = a_s^2 - b_s^2, \qquad s \in S'. \tag{3}$$

Thus, if $f \in C(S)$, we have

$$\int_S f dm_2 + \Sigma_{s \in S'}(a_s^1 + b_s^1)f(s) = \int_S f dm_2 + \Sigma_{s \in S'}(a_s^2 + b_s^2)f(s).$$

This together with (1) and $m_1(\{s\}) = m_2(\{s\}) = 0$, $s \in S'$, gives

$$m_1 = m_2; \ a_s^1 + b_s^1 = a_s^2 + b_s^2, \quad s \in S'.$$

By (3) we infer $a_s^1 = a_s^2, b_s^1 = b_s^2$, hence $\mu_1 = \mu_2$ and the proof that K is a simplex is completed.

Now define $\chi : S \to K$ by $\chi(s) = (s,0)$ and consider the operator $T: A(K) \to C(S)$ given by

$$T(g)(s) = g(\chi(s)) = g(s,0), \quad g \in A(K), \ s \in S.$$

Clearly $T \geqq 0$, $T1 = 1$. We are going to show that T is an extreme positive operator despite the fact that $\chi(s)$ is not an extreme point of K whenever $s \in S'$. If T were not an extreme positive operator then there would exist a non-identically null w^*-continuous function $\psi : S \to A^*(K)$ such that $\chi(s) \pm \psi(s) \in K$ for each $s \in S$. If $s \in S - S'$ then $\psi(s) = 0$, since $\chi(s)$ is an extreme point of K. Now let $s \in S'$. Since K is a simplex and $\chi(s)$ is the middle of the segment joining the extreme points (s, e_s) $(s, -e_s)$ we have $\chi(s) + \psi(s) = (s, \lambda_s e_s)$ where $|\lambda_s| \leqq 1$. Choose a net $\{s_\alpha\} \subset S'$, $s_\alpha \to s$, $s_\alpha \neq s$. Then $\chi(s_\alpha) + \psi(s_\alpha) \to \chi(s) + \psi(s)$ and, on the other hand, $(s_\alpha, \lambda_{s_\alpha} e_{s_\alpha}) \to (s,0)$. Hence $\lambda_s = 0$ and $\psi(s) = 0$. We proved that $\psi \equiv 0$, in other words T is an extreme positive operator.

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